Problem 1.54

Check the divergence theorem for the function

$$\mathbf{v} = r^2\cos\theta\,\hat{\mathbf{r}} + r^2\cos\phi\,\hat{\boldsymbol{\theta}} - r^2\cos\theta\sin\phi\,\hat{\boldsymbol{\phi}},$$

using as your volume one octant of the sphere of radius R (Fig. 1.48). Make sure you include the *entire* surface. [*Answer:* $\pi R^4/4$.]

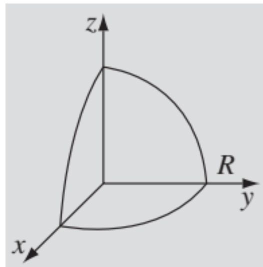


Fig. 1.48

Solution

In spherical coordinates (r,ϕ,θ), where θ is the angle from the polar axis, the divergence of a vector function is

$$\nabla\cdot\mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(v_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}.$$

For the given function, it evaluates to

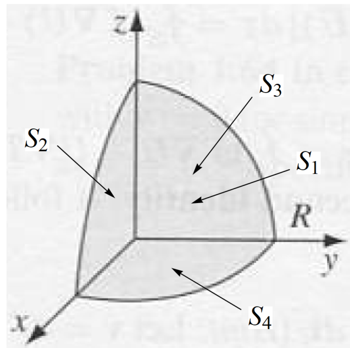
$$\begin{aligned}\nabla\cdot\mathbf{v} &= \frac{1}{r^2}\frac{\partial}{\partial r}[r^2(r^2\cos\theta)] + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}[(r^2\cos\phi)\sin\theta] + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(-r^2\cos\theta\sin\phi)\\ &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^4\cos\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(r^2\cos\phi\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(-r^2\cos\theta\sin\phi)\\ &= \frac{1}{r^2}(4r^3\cos\theta) + \cancel{\frac{1}{r\sin\theta}(r^2\cos\phi\cos\theta)} + \cancel{\frac{1}{r\sin\theta}(-r^2\cos\theta\cos\phi)}\\ &= 4r\cos\theta.\end{aligned}$$

The divergence theorem (or Gauss's theorem) relates the volume integral of $\nabla\cdot\mathbf{v}$ to a closed surface integral.

$$\iiint_D \nabla\cdot\mathbf{v}\,dV = \oiint_{\text{bdy } D} \mathbf{v}\cdot d\mathbf{S}$$

If D is the octant shown in Fig. 1.48 and $\mathbf{v} = r^2\cos\theta\,\hat{\mathbf{r}} + r^2\cos\phi\,\hat{\boldsymbol{\theta}} - r^2\cos\theta\sin\phi\,\hat{\boldsymbol{\phi}}$, then the left side becomes

$$\begin{aligned}\iiint_D \nabla\cdot\mathbf{v}\,dV = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^R (4r\cos\theta)(r^2\sin\theta\,dr\,d\phi\,d\theta) &= 2\left(\int_0^{\pi/2}\sin 2\theta\,d\theta\right)\left(\int_0^{\pi/2}d\phi\right)\left(\int_0^R r^3\,dr\right)\\ &= 2(1)\left(\frac{\pi}{2}\right)\left(\frac{R^4}{4}\right) = \frac{\pi R^4}{4}.\end{aligned}$$



Labelling the boundary of D as shown above, the right side evaluates to

$$\oiint_{\text{bdy } D} \mathbf{v}\cdot d\mathbf{S} = \iint_{S_1} \mathbf{v}\cdot d\mathbf{S} + \iint_{S_2} \mathbf{v}\cdot d\mathbf{S} + \iint_{S_3} \mathbf{v}\cdot d\mathbf{S} + \iint_{S_4} \mathbf{v}\cdot d\mathbf{S}$$

$$= \int_0^{\pi/2}\int_0^{\pi/2} [r^2\cos\theta\,\hat{\mathbf{r}} + r^2\cos\phi\,\hat{\boldsymbol{\theta}} - r^2\cos\theta\sin\phi\,\hat{\boldsymbol{\phi}}]\Big|_{r=R} \cdot (\hat{\mathbf{r}}R^2\sin\theta\,d\phi\,d\theta)$$

$$+ \int_0^{\pi/2}\int_0^{R} [r^2\cos\theta\,\hat{\mathbf{r}} + r^2\cos\phi\,\hat{\boldsymbol{\theta}} - r^2\cos\theta\sin\phi\,\hat{\boldsymbol{\phi}}]\Big|_{\phi=0} \cdot (-\hat{\boldsymbol{\phi}}r\,dr\,d\theta)$$

$$+ \int_0^{\pi/2}\int_0^{R} [r^2\cos\theta\,\hat{\mathbf{r}} + r^2\cos\phi\,\hat{\boldsymbol{\theta}} - r^2\cos\theta\sin\phi\,\hat{\boldsymbol{\phi}}]\Big|_{\phi=\pi/2} \cdot (\hat{\boldsymbol{\phi}}r\,dr\,d\theta)$$

$$+ \int_0^{\pi/2}\int_0^{R} [r^2\cos\theta\,\hat{\mathbf{r}} + r^2\cos\phi\,\hat{\boldsymbol{\theta}} - r^2\cos\theta\sin\phi\,\hat{\boldsymbol{\phi}}]\Big|_{\theta=\pi/2} \cdot (\hat{\boldsymbol{\theta}}r\,dr\,d\phi)$$

$$= \int_0^{\pi/2}\int_0^{\pi/2} [R^2\cos\theta\,\hat{\mathbf{r}} + R^2\cos\phi\,\hat{\boldsymbol{\theta}} - R^2\cos\theta\sin\phi\,\hat{\boldsymbol{\phi}}] \cdot (\hat{\mathbf{r}}R^2\sin\theta\,d\phi\,d\theta)$$

$$+ \int_0^{\pi/2}\int_0^{R} [r^2\cos\theta\,\hat{\mathbf{r}} + r^2(1)\,\hat{\boldsymbol{\theta}} - (0)\,\hat{\boldsymbol{\phi}}] \cdot (-\hat{\boldsymbol{\phi}}r\,dr\,d\theta)$$

$$+ \int_0^{\pi/2}\int_0^{R} [r^2\cos\theta\,\hat{\mathbf{r}} + (0)\,\hat{\boldsymbol{\theta}} - r^2\cos\theta(1)\,\hat{\boldsymbol{\phi}}] \cdot (\hat{\boldsymbol{\phi}}r\,dr\,d\theta)$$

$$+ \int_0^{\pi/2}\int_0^{R} [(0)\,\hat{\mathbf{r}} + r^2\cos\phi\,\hat{\boldsymbol{\theta}} - (0)\,\hat{\boldsymbol{\phi}}] \cdot (\hat{\boldsymbol{\theta}}r\,dr\,d\phi)$$

$$= \int_0^{\pi/2}\int_0^{\pi/2} R^4\sin\theta\cos\theta\,d\phi\,d\theta - \cancel{\int_0^{\pi/2}\int_0^{R} r^3\cos\theta\,dr\,d\theta} + \cancel{\int_0^{\pi/2}\int_0^{R} r^3\cos\phi\,dr\,d\phi}$$

$$= R^4\left(\int_0^{\pi/2} d\phi\right)\left(\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta\right)$$

$$= R^4\left(\frac{\pi}{2}\right)\left(\frac{1}{2}\right)$$

$$= \frac{\pi R^4}{4}.$$